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Resonances in twisted quantum waveguides

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Abstract

In this paper we consider embedded eigenvalues of a Schrödinger Hamiltonian in a waveguide induced by a symmetric perturbation. It is shown that these eigenvalues become unstable and turn into resonances after twisting of the waveguide. The perturbative expansion of the resonance width is calculated for weakly twisted waveguides and the influence of the twist on resonances in a concrete model is discussed in detail.

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1. Introduction

Quantum and electromagnetic waveguides have been studied since many decades; see [18, 22, 23, 27] and also [24] where the classical and the quantum pictures are compared. In this framework the spectral analysis of differential operators in tubular domains has become a research field of a certain interest [6, 15, 16]. Moreover, with the introduction of nano-devices, such as nanotubes, new open problems in quantum transmission for such structures appeared [7].

We consider here a waveguide type domain $\Omega = \mathbb{R} \times \omega$ (see figure 1, on the left), where the cross section ω of the waveguide is an open bounded and connected subset of $\mathbb{R}^2$. We impose Dirichlet boundary conditions at the boundary of Ω. The spectrum of the free operator $-\Delta$ in $L^2(\Omega)$ is absolutely continuous and covers the half-line $[E_1, \infty)$, where E_1 is the lowest eigenvalue of the Dirichlet Laplacian on ω. It is a well-known fact that this spectrum is unstable against perturbations; indeed, a negative perturbation, vanishing at infinity, of $-\Delta$ will induce at least one bound state below the threshold E_1 and new embedded bound states in the half-line $[E_1, +\infty)$ (see, e.g., figure 2). The perturbation can be either of a potential type or of a geometric type; see [5, 9, 16] and references there. Similar effects occur also in two-dimensional strips [2, 4, 15]. These new bound states below the threshold E_1 correspond

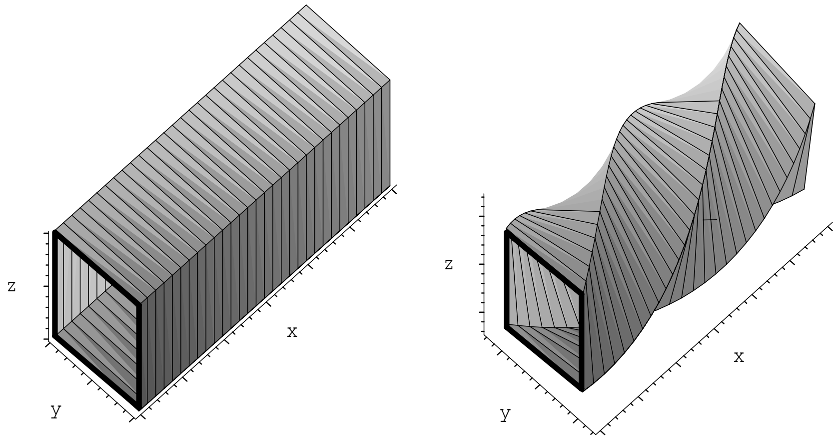

Figure 1. On the left, a plot of the surface of a rectangular waveguide without twisting; on the right, a plot of the surface of the twisted rectangular waveguide. The bold line represents the boundary of ω.

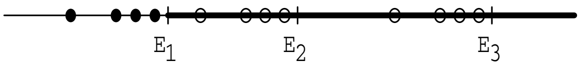

Figure 2. The discrete spectrum of H_0^V consists of finitely many simple eigenvalues below E_1 (denoted by a full circle); the essential spectrum is given by the half-line $[E_1, +\infty)$. Furthermore, a non-empty set of simple eigenvalues (denoted by an empty circle) embedded in the half-line $[E_1, +\infty)$ occurs.

to the particles (electrons) which do not propagate along Ω, but remain localized in a bounded region of Ω.

Recently it has been shown [14] that the presence of bound states in Ω can be, up to a certain extent, suppressed by another geometrical perturbation: the so-called *twisting* which is defined as follows. For a given $x \in \mathbb{R}$ and $s := (y, z) \in \omega$, we define the mapping

$$f_\varepsilon : \mathbb{R} \times \omega \to \mathbb{R}^3$$

by

$$f_\varepsilon(x, s) = (x, y\cos(\varepsilon\alpha(x)) + z\sin(\varepsilon\alpha(x)), z\cos(\varepsilon\alpha(x)) - y\sin(\varepsilon\alpha(x))), \tag{1}$$

where $\varepsilon > 0$ is a real parameter and $\alpha : \mathbb{R} \to \mathbb{R}$ is a differentiable function. Furthermore, we introduce

$$\Omega_\varepsilon := f_\varepsilon(\Omega).$$

Clearly, Ω_ε is a tube which is twisted unless the function α is constant (in figure 1 we plot, respectively, a rectangular tube without and with twisting). The result of [14] shows that if the cross section ω is not rotationally symmetric and the tube Ω is twisted, even only locally, then the bound states for the perturbed Hamiltonian $-\Delta + V$ do not appear for any negative potential $V(x)$, but only if V is strong enough. In other words, one could say that a twisting of a tubular domain Ω improves the transport of charged particles in Ω in the sense that it protects the particles to get trapped by weak perturbations. The repulsive effect of twisting has been recently observed also in [17], where the absence of discrete eigenvalues in tubes which are simultaneously mildly curved and mildly twisted. Moreover, in [3] the repulsive effect of twisting is demonstrated for bounded tubes whose thickness goes to zero.

Similar results were also obtained for two-dimensional waveguides with combined boundary conditions or with a local magnetic field [13, 20]. However, the geometrical perturbations of the waveguide generically induce also the existence of *resonances*, i.e. metastable states with very long lifetimes (see [10–12]).

It is the aim of the present paper to describe the influence of twisting on the resonances in the waveguides. More precisely, we study the situation in which the free Laplacian is perturbed by an attractive potential $V(x)$, which decays at infinity along the waveguide direction, where $x \in \mathbb{R}$ represents the coordinate along the waveguide direction. The point spectrum of the perturbed Hamiltonian $-\Delta + V(x)$ consists, in addition to the bound states below E_1, of infinitely many eigenvalues embedded in the continuum $[E_1, \infty)$ (see figure 2). It was shown in [10], for two-dimensional waveguides, that these embedded eigenvalues generically turn into resonances in the presence of a constant magnetic field. Following the method of [10], we show that this happens also when the magnetic field is replaced by the twisting, provided the cross section ω is not rotationally symmetric (see theorem 1). For weak twisting we also give the perturbative expansion of the corresponding resonance width.

In order to obtain a precise estimate on the imaginary part of the resonances and, in particular, to prove that it is strictly negative we consider in section 5 a concrete model in which the potential V approximates a one-dimensional point interaction. For such a model we explicitly calculate the leading term of the imaginary part of a chosen resonance, see proposition 1, and we prove that for suitable values of the parameters (see remark 9), the imaginary part of the resonance is strictly negative.

2. Preliminaries

Throughout the paper we will denote by $\langle \cdot, \cdot \rangle_H$ the scalar product in a Hilbert space H with the convention $\langle \alpha u, v\rangle_H = \bar{\alpha}\langle u, v\rangle_H$ for all $\alpha \in \mathbb{C}$ and $u, v \in H$. For a real-valued measurable bounded function $V(x)$ on $\mathbb{R}$ we formally define the Hamiltonians

$$\tilde{H}^0_\varepsilon = -\Delta \qquad \text{and} \qquad \tilde{H}^V_\varepsilon = -\Delta + V(x) \qquad \text{in} \qquad L^2(\Omega_\varepsilon)$$

with Dirichlet boundary conditions at $\partial\Omega_\varepsilon$. The operator $\tilde{H}^V_\varepsilon$ is associated with the closed quadratic form

$$\tilde{Q}^V_\varepsilon[\psi] := \int_{\Omega_\varepsilon} [|\nabla\psi|^2 + V(x)|\psi|^2]\,\mathrm{d}x\,\mathrm{d}s, \tag{2}$$

with the form domain $D\big(\tilde{Q}^V_\varepsilon\big) = \mathcal{H}^1_0(\Omega_\varepsilon)$.

Given a test function $\psi \in C_0^\infty(\Omega)$ it is useful to introduce the following shorthand:

$$\partial_\tau\psi := y\partial_z\psi - z\partial_y\psi. \tag{3}$$

As usual in such situations, in order to analyse the operator $\tilde{H}^V_\varepsilon$ we pass from the twisted tube Ω_ε to the untwisted tube Ω by means of a simple substitution of variables. This gives

$$Q^V_\varepsilon[\psi] = \int_\Omega (|\nabla_s\psi|^2 + |\partial_x\psi + \varepsilon\dot{\alpha}(x)\partial_\tau\psi|^2 + V(x)|\psi|^2)\,\mathrm{d}x\,\mathrm{d}s,$$

with the form domain $D\big(Q^V_\varepsilon\big) = \mathcal{H}^1_0(\Omega)$ and with the notation

$$\nabla_s\psi := (\partial_y\psi, \partial_z\psi).$$

In other words, the operator H^V_ε, associated with Q^V_ε and unitarily equivalent to $\tilde{H}^V_\varepsilon$, acts on its domain in $L^2(\Omega)$ in the weak sense as

$$H^V_\varepsilon = -\partial_y^2 - \partial_z^2 - [\partial_x + \varepsilon\dot{\alpha}(x)\partial_\tau]^2 + V(x) = H^V_0 + U^V_\varepsilon,$$

where

$$H_0^V = -\partial_x^2 - \partial_y^2 - \partial_z^2 + V(x)$$

and

$$\begin{aligned} U_\varepsilon^V &= -[\partial_x + \varepsilon\dot{\alpha}(x)\partial_\tau]^2 + \partial_x^2 \\ &= -2\varepsilon\dot{\alpha}(x)\partial_x\partial_\tau - \varepsilon\ddot{\alpha}(x)\partial_\tau - \varepsilon^2\dot{\alpha}^2(x)\partial_\tau^2. \end{aligned}$$

Remark 1. The term U_ε^V is a symmetric operator on $L^2(\Omega)$ with Dirichlet boundary conditions at $\partial\Omega$.

In order to show that the embedded eigenvalues of H_0^V turn into the resonances once the waveguide is twisted, we employ the method of the exterior complex scaling in combination with the regular perturbation theory [8]. We start by locating the spectrum of the untwisted model.

3. Spectrum of H_0^V

We will suppose that V satisfies the following

Assumption A. The function $V(x)$ is not identically equal to zero and

$$\int_{\mathbb{R}} (1 + x^2)|V(x)|\,\mathrm{d}x < \infty \qquad \text{and} \qquad \int_{\mathbb{R}} V(x)\,\mathrm{d}x \leqslant 0. \tag{4}$$

It then follows from [25] (see, e.g. theorem XIII.110 in and its Notes) that the operator

$$h := -\partial_x^2 + V(x) \qquad \text{in} \quad L^2(\mathbb{R})$$

possesses finitely many negative eigenvalues $\{\mu_j\}_{j=1}^N$, $N \geqslant 1$, each of multiplicity 1. We denote by $\varphi_j(x)$ the corresponding normalized eigenfunctions. The essential spectrum of h covers the positive half-line $[0, \infty)$. On the other hand, it is well known that the operator $-\Delta_D^\omega$, i.e. the Dirichlet Laplacian on ω, is positive definite and has purely discrete spectrum. Let $\{E_n\}_{n=1}^\infty$ be the non-decreasing sequence of its eigenvalues and let $\chi_n(s)$ denote the associated normalized eigenfunctions. The set of such eigenfunctions is an orthonormal basis of $L^2(\omega)$. We denote by

$$\Sigma = \{E = \mu_j + E_n,\ j = 1, \ldots, N, n \geqslant 1\}$$

the set of eigenvalues of H_0^V with associated normalized eigenvectors

$$\psi_{n,j}(x, s) = \varphi_j(x)\chi_n(s)$$

and

$$\Sigma_+ = \Sigma \cap [E_1, +\infty), \qquad \Sigma_- = \Sigma \cap (-\infty, E_1),$$

where Σ_- is not empty since $\mu_j < 0$ for any j. Then, by the standard arguments [25] the spectrum of

$$H_0^V = -\Delta + V(x), \qquad \text{in} \quad L^2(\mathbb{R} \times \omega)$$

is given by $\sigma\left(H_0^V\right) = \sigma_d\left(H_0^V\right) \cup \sigma_{\text{ess}}\left(H_0^V\right)$, where

$$\sigma_d\left(H_0^V\right) = \Sigma_- \qquad \text{and} \qquad \sigma_{\text{ess}}\left(H_0^V\right) = [E_1, \infty).$$

In addition, H_0^V possesses a point spectrum embedded into the continuum given by Σ_+ (see figure 2).

We expect that when ε becomes non-zero these embedded eigenvalues generically turn into resonances, which are the main object of our study.

Remark 2. Since the operator H_0^V commutes with complex conjugation, its eigenfunctions ψ can be assumed to be real-valued.

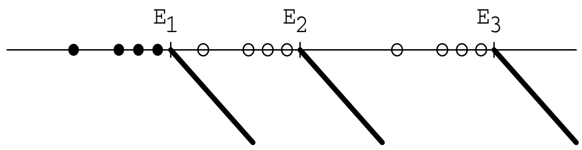

Figure 3. The discrete spectrum of $H_0^V(\theta)$ consists of a sequence of real and simple eigenvalues (denoted by a circle); the essential spectrum is given by the half-lines $E_n + \mathrm{e}^{-2\mathrm{i}\,\mathrm{Im}\,\theta}\mathbb{R}^+$.

4. Complex scaling

Henceforth, we will employ the method of exterior complex scaling to the operator H_ε^V. In order to do so, we will need some assumptions on the functions V and α:

Assumption B. V extends to an analytic function with respect to x in some sector

$$M_\beta := \{\zeta \in \mathbb{C} : |\arg \zeta| \leqslant \beta\}, \qquad \text{with} \quad \beta > 0.$$

Moreover, V is uniformly bounded in M_β.

Assumption C. α extends to analytic function with respect to x in

$$\mathcal{M}_\beta = M_\beta \cup \{\zeta \in \mathbb{C} : |\mathrm{Im}\,\zeta| \leqslant \beta\}, \qquad \text{with} \quad \beta > 0,$$

and $\dot\alpha$ is uniformly bounded in $\mathcal{M}_\beta$. In addition, $\dot\alpha(x) > 0, \forall x \in \mathbb{R}$.

Remark 3. Since $\dot\alpha$ is uniformly bounded in $\mathcal{M}_\beta$, from the Cauchy theorem it follows that $\ddot\alpha$ is uniformly bounded in $\mathcal{M}_{\beta'}$ for any $0 < \beta' < \beta$.

In analogy with [10] we introduce the mapping S_θ, which acts as a complex dilation in the longitudinal variable x:

$$(S_\theta \psi)(x, s) = \mathrm{e}^{\theta/2}\psi(\mathrm{e}^\theta x, s), \qquad \theta \in \mathbb{C}.$$

The transformed operator then takes the form

$$H_\varepsilon^V(\theta) = S_\theta H_\varepsilon^V S_\theta^{-1} = H_0^V(\theta) + U_\varepsilon^V(\theta),$$

where

$$H_0^V(\theta) = S_\theta H_0^V S_\theta^{-1} = -\mathrm{e}^{-2\theta}\partial_x^2 - \partial_y^2 - \partial_z^2 + V(\mathrm{e}^\theta x)$$

and

$$U_\varepsilon^V(\theta) = S_\theta U_\varepsilon^V S_\theta^{-1} = -2\varepsilon\,\mathrm{e}^{-\theta}\dot\alpha(e^\theta x)\partial_x\partial_\tau - \varepsilon\,\mathrm{e}^{-\theta}\ddot\alpha(e^\theta x)\partial_\tau - \varepsilon^2\dot\alpha^2(e^\theta x)\partial_\tau^2. \tag{5}$$

Lemma 1. *Let V satisfy assumptions A and B, then $H_0^V(\theta)$ is an analytic family of type* A *with respect to θ. Furthermore, the spectrum of $H_0^V(\theta)$ has the form (see figure 3)*

$$\sigma\left(H_0^V(\theta)\right) = \bigcup_n [E_n + \mathrm{e}^{-2\mathrm{i}\,\mathrm{Im}\,\theta}\mathbb{R}^+]. \tag{6}$$

More precisely, the essential spectrum of $H_0^V(\theta)$ consists of the sequence of the half-lines $E_n + \mathrm{e}^{-2\mathrm{i}\,\mathrm{Im}\,\theta}\mathbb{R}^+$, $n = 1, 2, \ldots,$ and the discrete spectrum of $H_0^V(\theta)$ consists of the set of eigenvalues $\mu_j + E_n$ with associated eigenvectors

$$[\psi_{n,j}(\theta)](x, s) = [S_\theta\psi_{n,j}](x, s) = \mathrm{e}^{\theta/2}\varphi_j(e^\theta x)\chi_n(s). \tag{7}$$

Proof. It follows from assumption B that the family of operators $H_0^V(\theta)$ is analytic of type A with respect to θ (see [19, chapter 7]). For what concerns its spectrum it is enough to remark that the operator

$$h(\theta) = S_\theta h S_\theta^{-1} = -\mathrm{e}^{-2\theta}\partial_x^2 + V(\mathrm{e}^\theta x)$$

in $L^2(\mathbb{R})$ has the spectrum given by

$$\sigma(h(\theta)) = \{\mu_1, \ldots, \mu_N\} \cup \mathrm{e}^{-2\mathrm{i}\,\mathrm{Im}\,\theta}\mathbb{R}^+.$$

By the Ichinose's lemma, see [25, Sec. XIII.10], we then obtain formula (6) for the spectrum of the sum $h(\theta) - \partial_y^2 - \partial_z^2$. □

Lemma 2. *Let V satisfy assumptions A and B and let α satisfy assumption C, then the operator $U_\varepsilon^V(\theta)$ is a relatively bounded perturbation of $H_0^V(\theta)$. Moreover, the family of operators $H_\varepsilon^V(\theta)$ is analytic of type A for all θ such that $|\theta| < R_\varepsilon$, where $R_\varepsilon \to \infty$ as $\varepsilon \to 0$.*

Proof. In order to prove the lemma we consider a test function $\psi \in C_0^\infty(\Omega)$. Using the assumptions on V we find out that there exists a positive constant c (here and below, c will denote a positive constant whose value changes from line to line) such that

$$\left\|H_0^V(\theta)\psi\right\|^2 \geqslant c\left(\left\|\partial_x^2\psi\right\|^2 + \left\|\partial_y^2\psi\right\|^2 + \left\|\partial_z^2\psi\right\|^2\right) - c\|\psi\|^2.$$

From this inequality and using the fact that ω is bounded we arrive at

$$\begin{aligned}\left\|\partial_\tau^2\psi\right\|^2 &\leqslant c\left(\left\|\partial_z^2\psi\right\|^2 + \left\|\partial_y^2\psi\right\|^2 + \|\partial_z\psi\|^2 + \|\partial_y\psi\|^2\right)\\ &\leqslant c\left(\left\|\partial_z^2\psi\right\|^2 + \left\|\partial_y^2\psi\right\|^2 + \|\psi\|^2\right)\\ &\leqslant c\left(\left\|H_0^V(\theta)\psi\right\|^2 + \|\psi\|^2\right)\end{aligned}$$

and

$$\|\partial_\tau\psi\|^2 \leqslant c(\|\partial_z\psi\|^2 + \|\partial_y\psi\|^2) \leqslant c\left(\left\|H_0^V(\theta)\psi\right\|^2 + \|\psi\|^2\right). \tag{8}$$

As for the mixed term in $U_\varepsilon^V(\theta)$ we note that

$$\|\partial_x\partial_\tau\psi\|^2 = \langle\partial_x\partial_\tau\psi, \partial_x\partial_\tau\psi\rangle_{L^2(\Omega)} \leqslant \left\|\partial_\tau^2\psi\right\|\left\|\partial_x^2\psi\right\| \leqslant \tfrac{1}{2}\left[\left\|\partial_\tau^2\psi\right\|^2 + \left\|\partial_x^2\psi\right\|^2\right]. \tag{9}$$

Collecting all these estimates we finally conclude that there exists a positive constant C, depending on ω and V, such that

$$\left\|U_\varepsilon^V(\theta)\psi\right\|^2 \leqslant C(\varepsilon + \varepsilon^2)^2\left(\left\|H_0^V(\theta)\psi\right\|^2 + \|\psi\|^2\right). \tag{10}$$

To prove the second statement of the lemma we first notice that by assumption B we have $D\left(H_0^V(\theta)\right) = D\left(H_0^V(0)\right)$. By assumption C and [19, section 7.2] it thus suffices to show that both $\partial_x\partial_\tau$ and ∂_τ are relatively bounded with respect to $H_0^V(\theta)$. However, this follows from (8) and (10). □

Lemma 2 tells us that the eigenvalues of $H_\varepsilon^V(\theta)$ are analytic functions of θ. By a standard argument [8], it turns out that they are in fact independent of θ. The non-real eigenvalues of $H_\varepsilon^V(\theta)$, for θ such that $\mathrm{Im}\,\theta > 0$, are identified with the resonances of H_ε^V [8].

Remark 4. As a result of the previous proof it follows that $U_\varepsilon^V(\theta)$ is a regular perturbation of the operator $H_0^V(\theta)$. This enables us to apply the analytic perturbation theory to the eigenvalues of the operator $H_0^V(\theta)$.

Theorem 1. *Let $E = E_n + \mu_j \in \Sigma_+$ be a simple eigenvalue of $H_0^V\theta)$ where θ is a fixed complex number such that $\mathrm{Im}\,\theta > 0$. For any ball B centred in E there exists $\varepsilon^\star > 0$, depending on E,* *such that for any* ε *with* $|\varepsilon| < \varepsilon^\star$*, there is an eigenvalue* $E(\varepsilon)$ *of* $H_\varepsilon^V(\theta)$ *belonging to* B *and with the imaginary part given by*

$$\operatorname{Im} E(\varepsilon) = -\varepsilon^2 a + O(\varepsilon^3) \tag{11}$$

where a *is a constant independent of* ε *and equal to*

$$a = \sum_{k\leqslant k^\star} |\langle \partial_\tau \chi_n, \chi_k\rangle_{L^2(\omega)}|^2 \langle v_j, \operatorname{Im} \hat{r}(E - E_k) v_j\rangle_{L^2(\mathbb{R})}. \tag{12}$$

Here

$$v_j = (-2\dot{\alpha}\partial_x + \ddot{\alpha})\varphi_j, \qquad k^\star = \max\{k : E_k - E < 0\}$$

and $\operatorname{Im}\hat{r}$ *stands for the imaginary part of the reduced resolvent [19] of* $h = -\partial_x^2 + V$ *with respect to the eigenvalue* μ_j*:*

$$\langle v_j, \operatorname{Im}\hat{r}(E - E_k)v_j\rangle_{L^2(\mathbb{R})} = \lim_{\rho\to 0^+} \frac{1}{2\mathrm{i}}[\langle v_j, \hat{r}(E - (E_k + \mathrm{i}\rho))v_j\rangle_{L^2(\mathbb{R})}$$
$$- \langle v_j, \hat{r}(E - (E_k - \mathrm{i}\rho))v_j\rangle_{L^2(\mathbb{R})}].$$

Proof. Let $\psi(\theta) = \psi_{n,j}(\theta)$ be the associated normalized eigenvector (7) belonging to E, where n and j are fixed. We apply the regular perturbation theory saying that for some fixed $r > 0$ small enough and for any ε with modulus small enough, in the given ball $B_r(E)$ exists only one eigenvalue $E(\varepsilon)$ of $H_\varepsilon^V(\theta)$ with an associated eigenvector

$$\psi^\varepsilon(\theta) = \frac{1}{2\pi\mathrm{i}} \oint_{\partial B_r} \left[\zeta - H_\varepsilon^V(\theta)\right]^{-1} \psi(\theta)\,\mathrm{d}\zeta.$$

Furthermore, the regular perturbation theory also yields that this eigenvalue is given by means of the convergent perturbative series (see, e.g. [25, XII.6])

$$E(\varepsilon) = \frac{\left\langle \bar{\psi}(\theta), H_\varepsilon^V(\theta)\psi^\varepsilon(\theta)\right\rangle_{L^2(\Omega)}}{\langle \bar{\psi}(\theta), \psi^\varepsilon(\theta)\rangle_{L^2(\Omega)}} = \sum_{m=0}^{\infty} e_m(\varepsilon), \qquad e_m = O(\varepsilon^m)$$

where, as usual,

$$e_0 = E \qquad \text{and} \qquad e_1 = \frac{\left\langle \bar{\psi}(\theta), U_\varepsilon^V(\theta)\psi(\theta)\right\rangle_{L^2(\Omega)}}{\langle \bar{\psi}(\theta), \psi(\theta)\rangle_{L^2(\Omega)}} = \frac{\left\langle \psi, U_\varepsilon^V \psi\right\rangle_{L^2(\Omega)}}{\langle \psi, \psi\rangle_{L^2(\Omega)}}$$

are constant with respect to θ, and ψ is the real-valued vector (7) for $\theta = 0$ (see remark 2). These constants e_0 and e_1 are real-valued since U_ε^V is a symmetric operator (see remark 1). If we prove that $\operatorname{Im} e_2 = -\varepsilon^2 a + O(\varepsilon^3)$ for some $a > 0$ independent of ε then the stated result follows. To this end, we recall that (see [10])

$$\operatorname{Im} e_2 = \operatorname{Im} a_2(1 + O(\varepsilon)),$$

where

$$a_2 = -\frac{1}{2\pi\mathrm{i}} \oint_{\partial B_r} \left\langle \bar{\psi}(\theta), U_\varepsilon^V(\theta)\left[\zeta - H_0^V(\theta)\right]^{-1} U_\varepsilon^V(\theta)\psi(\theta)\right\rangle_{L^2(\Omega)} \frac{\mathrm{d}\zeta}{\zeta - E}$$
$$= \lim_{\rho\to 0^+} f(\theta, E + \mathrm{i}\rho) = \lim_{\rho\to 0^+} f(\theta = 0, E + \mathrm{i}\rho)$$

and

$$f(\theta, \zeta) = -\left\langle \bar{\psi}(\theta), U_\varepsilon^V(\theta)\left[\zeta - H_\varepsilon^V(\theta)\right]^{-1} U_\varepsilon^V(\theta)\psi(\theta)\right\rangle_{L^2(\Omega)}$$
$$+ \left|\left\langle \bar{\psi}(\theta), U_\varepsilon^V(\theta)\psi(\theta)\right\rangle_{L^2(\Omega)}\right|^2 (\zeta - E)^{-1}.$$

Hence

$$a_2 = -\langle\psi, U_\varepsilon^V \hat{R}(E+\mathrm{i}0)U_\varepsilon^V\psi\rangle_{L^2(\Omega)} := \lim_{\rho\to 0^+}\left[-\langle\psi, U_\varepsilon^V \hat{R}(E+\mathrm{i}\rho)U_\varepsilon^V\psi\rangle_{L^2(\Omega)}\right] \tag{13}$$

where $\hat{R}(\zeta)$ is the reduced resolvent of H_0^V with respect to the eigenvalue E (see [25, X.II.6] and [19]). Recalling that ψ has the form

$$\psi(x,s) = \psi_{n,j}(x,s) = \varphi_j(x)\chi_n(s)$$

for some n and j and that $\{\chi_k(s)\}_{k=1}^\infty$ is a basis of $L^2(\omega)$, we obtain

$$U_\varepsilon^V\psi(x,s) = \sum_{k=1}^\infty d_k(x)\chi_k(s), \qquad \text{where} \quad d_k(x) = \langle\chi_k, U_\varepsilon^V\psi\rangle_{L^2(\omega)}.$$

Using the fact that $U_\varepsilon^V\psi \in L^2(\Omega)$, lemma 2 and the dominated convergence theorem, we conclude that

$$a_2 = -\sum_{k=1}^\infty \langle d_k, \hat{r}(E-E_k+\mathrm{i}0)d_k\rangle_{L^2(\mathbb{R})}, \tag{14}$$

where $\hat{r}(\zeta)$ is the reduced resolvent of $h = -\partial_x^2 + V$ with respect to μ_j. Concerning the imaginary part of a_2 we point out that only finitely many terms on the rhs of (14) have a non-zero imaginary part. The latter follows from the fact that $\langle d_k, \hat{r}(E-E_k+\mathrm{i}0)d_k\rangle$ is real for any k large enough, more precisely for any $k > k^\star$, where

$$k^\star = \max\{k : E_k - E < 0\}.$$

From

$$d_k(x) = \varepsilon v_j(x)\langle\partial_\tau\chi_n, \chi_k\rangle_{L^2(\omega)}[1+O(\varepsilon)], \qquad v_j = (-2\dot{\alpha}\partial_x + \ddot{\alpha})\varphi_j,$$

we can thus conclude that

$$a_2 = -\varepsilon^2 A[1+O(\varepsilon)],$$

where

$$A = A_{n,j} = \sum_{k\leqslant k^\star} |\langle\partial_\tau\chi_n, \chi_k\rangle_{L^2(\omega)}|^2 \langle v_j, \hat{r}(E-E_k+\mathrm{i}0)v_j\rangle_{L^2(\mathbb{R})}$$

is independent of ε. This implies (12) since

$$a = \operatorname{Im} A = \sum_{k\leqslant k^\star} |\langle\partial_\tau\chi_n, \chi_k\rangle_{L^2(\omega)}|^2 \langle v_j, \operatorname{Im}\hat{r}(E-E_k)v_j\rangle_{L^2(\mathbb{R})}. \qquad \square$$

Remark 5. Note that if ω is rotationally symmetric, then $a = 0$. Indeed, since χ_n is rotationally symmetric whenever E_n is simple, this follows from (12).

Remark 6. We point out that $\operatorname{Im} r(\zeta)$ is a symmetric and positive operator for ζ real (see, e.g., [10]). We can thus *generically* expect that for any $\tilde{E} > E_1$ fixed there exists $\varepsilon^\star > 0$ small enough such that $H_\varepsilon^V(\theta)$ does not have a discrete spectrum in the interval $[E_1, \tilde{E}]$ for any $0 < |\varepsilon| \leqslant \varepsilon^\star$; more precisely, for any $\delta > 0$ the set

$$\sigma_d\big(H_\varepsilon^V(\theta)\big) \cap \{[E_1, \tilde{E}] \times \mathrm{i}[-\delta, +\delta]\}$$

is empty or it consists of finitely many points with the imaginary part strictly negative (see figure 4). As a result, it follows that the embedded eigenvalue $E \in \Sigma_+$ of the untwisted model turns into a resonance when an appropriate twisting is applied.

Concerning the assumption on the multiplicity of E we note that it is closely related to the multiplicity of E_n, the eigenvalues of $-\Delta_D^\omega$. In general, and especially for cross-sections

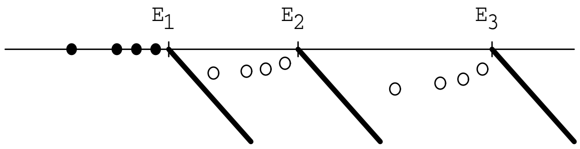

Figure 4. The discrete spectrum of $H_\varepsilon^V(\theta)$, for $\varepsilon \neq 0$ small enough, consists of two parts; the first part is given by the real and simple eigenvalues (full circle) below E_1, the second one is given by simple eigenvalues with real part larger that E_1 and with *imaginary part strictly negative* (empty circle).

with a high degree of symmetry, some eigenvalues of $-\Delta_D^\omega$ might be highly degenerated. However, this degeneracy is unstable under small perturbations of the domain [26]. In fact, if the boundary of ω can be C^k- smoothly embedded in $\mathbb{R}^2$ with $k > 4$, then the eigenvalues E_n of $-\Delta_D^\omega$ are *generically* simple (see [26]).

Remark 7. In [17, theorem 1] Grushin has obtained a similar asymptotic expansion for the ground state in mildly twisted tubes with transversal potential. He then proved, under certain assumptions on ω, the absence of discrete eigenvalues for ε small enough. We would like to mention that there is one important difference between our result and that of [14, 17]. Assume that the boundary of ω is sufficiently regular (e.g. C^1- smooth) and replace the Dirichlet boundary condition at $\partial\Omega$ by the *Neumann* one; let us denote the resulting operator by $H_\varepsilon^{V,N}$ Since V depends only on x, the eigenfunction associated with the lowest eigenvalue of $H_0^{V,N}$ will be given by $\psi_1(x, s) = c\varphi_1(x)$, where c is a constant. This follows from the fact that χ_1 is constant in the case of Neumann boundary conditions. Consequently we have $\partial_\tau\psi_1 \equiv 0$ and therefore, using ψ_1 as a test function, we find out that

$$\inf\sigma\left(H_\varepsilon^{V,N}\right) \leqslant \inf\sigma\left(H_0^{V,N}\right).$$

This means that, contrary to the Dirichlet case, the lowest eigenvalue of $H_0^{V,N}$ will not be removed by the twisting, even for very small V. On the other hand, the above analysis shows that the effect on the embedded eigenvalues will typically occur also in the Neumann case, since the eigenfunctions of $H_0^{V,N}$ associated with the embedded eigenvalues are not constant in s.

5. A concrete model

In the previous section we have seen that the embedded eigenvalues under the influence of twisting generically turn into resonances. However, theorem 1 does not *a priori* say that the imaginary part of the resonances is strictly negative. In this section, we will show on a concrete model that for mildly twisted waveguides one can guarantee the negativity of the imaginary part of a chosen resonance.

To make this problem simpler we would like to consider a concrete model, in which V acts as a Dirac delta potential. However, as the Dirac delta potential is obviously not dilation analytic, see assumption B, we will approximate it by the sequence

$$V_\nu(x) = -\frac{\nu}{2\cosh^2(\nu x)}, \qquad \nu > 0, \tag{15}$$

which converges to the delta function at zero as $\nu \to \infty$ in the sense of distributions. Moreover, to be able to give some quantitative results we assume that $\alpha(x) = x$.

Proposition 1. *Let $\alpha(x) = x$ and assume that the embedded eigenvalue $E = E_2 + \mu_1$ of the operator $H_0^{V_\nu}$ is simple and that $E_2 - E_1 > \frac{1}{4}$. Here V_ν is given by (15). Then in the vicinity of E there is an eigenvalue $E(\varepsilon, \nu)$ of $H_\varepsilon^{V_\nu}(\theta)$ with the imaginary part given by*

$$\operatorname{Im} E(\varepsilon, \nu) = -\varepsilon^2 a(\nu) + O(\varepsilon^3), \tag{16}$$

where

$$\lim_{\nu\to\infty} a(\nu) = |C_1|^2 \frac{\sqrt{E_2 - E_1 - \frac{1}{4}}}{(E_2 - E_1)^2}, \qquad C_1 = \langle \chi_1, \partial_\tau \chi_2 \rangle_{L^2(\omega)}. \tag{17}$$

In particular, if $C_1 \neq 0$ then the coefficient $a(\nu)$ is strictly positive for ν large enough.

Remark 8. We note that with this choice of α the essential spectrum of $H_\varepsilon^V(\theta)$ will depend on ε. This fact is not usual in complex scaling methods; however, this inconvenience does not affect the existence of the resonances since the correction of the essential spectrum is small for ε small and thus the non-real eigenvalues of $H_\varepsilon^V(\theta)$ will stay far enough from its essential spectrum. In fact, it could be possible to avoid this fact by choosing

$$\alpha(x) = x, \quad \forall x \in [-X, X] \qquad \text{and} \qquad \dot\alpha(x) = 0, \quad \forall |x| > 2X, \tag{18}$$

for some $X \gg 1$. We note that with this choice of α the essential spectrum of $H_\varepsilon^V(\theta)$ will not depend on ε and, furthermore, since the wavefunction φ_1 decays very fast as $|x|$ grows then, in order to compute (31) and (33), we can practically take $\dot\alpha = 1$ and $\ddot\alpha = 0$.

Remark 9. Equation (17) implies that if $C_1 \neq 0$, then the imaginary part $\operatorname{Im} E(\varepsilon, \nu)$ will be strictly negative for suitable values of ν end ε. Indeed, first we take ν large enough so that $a(\nu) > 0$. Then we keep this ν fixed and take $\varepsilon = \varepsilon(\nu)$ small enough so that the remainder term $O(\varepsilon^3)$, which also depends on ν, becomes smaller than $\varepsilon^2 a(\nu)$. Then

$$\operatorname{Im} E(\varepsilon, \nu) < 0, \tag{19}$$

which means that the twisting pushes the eigenvalue $E(\varepsilon, \nu)$ down in the complex plane, making thus the lifetime of the corresponding resonance shorter.

Remark 10. Note that the assumption $E_2 - E_1 > \frac{1}{4}$ is made only for the sake of simplicity. It guarantees that E is an embedded eigenvalue of $H_0^{V_\nu}$ for any positive ν. If $E_2 - E_1 < \frac{1}{4}$, then we would have to consider an eigenvalue $\tilde{E} = E_k + \mu_1$ for some k large enough such that $\tilde{E}$ is embedded.

Remark 11. The coefficient C_1 depends on the geometry of ω. An integration by parts shows that $C_1 = -\langle \partial_\tau \chi_1, \chi_2 \rangle_{L^2(\omega)}$. It is therefore easy to see that $C_1 = 0$ for rotationally symmetric ω. However, it is not *a priori* guaranteed that $C_1 \neq 0$ whenever ω is not rotationally symmetric, although we are not aware of any counter-example. Let us only mention that $C_1 \neq 0$ for certain domains ω. Indeed, for $\omega = [0, a] \times [0, b]$ with $a > b > 0$, we have

$$\chi_1(y, z) = \frac{2}{\sqrt{ab}} \sin\left(\frac{\pi}{a} y\right) \sin\left(\frac{\pi}{b} z\right), \qquad \chi_2(y, z) = \frac{2}{\sqrt{ab}} \sin\left(\frac{2\pi}{a} y\right) \sin\left(\frac{\pi}{b} z\right).$$

An explicit calculation then shows that

$$C_1 = \langle \chi_1, \partial_\tau \chi_2 \rangle_{L^2(\omega)} = -\frac{4b}{3a} \neq 0.$$

5.1. Proof of proposition 1

Equation (16) follows directly from theorem 1. The rest of the proof will be given in two steps.

Spectrum of $h_\nu = -\partial_x^2 + V_\nu$. Following [21, section 23] we set

$$t = \frac{1}{2}\left[-1 + \sqrt{1 + \frac{2}{\nu}}\right]. \tag{20}$$

The eigenvalue problem $h_\nu \tilde{\varphi}_j = \mu_j \tilde{\varphi}_j$ admits solutions

$$\mu_j = -\frac{\nu^2}{4}\left[-(2j-1) + \sqrt{1 + \frac{2}{\nu}}\right]^2, \qquad 1 \leqslant j < t + 1 \tag{21}$$

with associated eigenfunctions

$$\tilde{\varphi}_j(x) = (1 - \xi^2)^{e_j/2} F\left[e_j - t, e_j + t + 1, e_j + 1, \tfrac{1}{2}(1 - \xi)\right], \tag{22}$$

where

$$\xi = \tanh(\nu x), \qquad e_j = \frac{\sqrt{-\mu_j}}{\nu} = \frac{1}{2}\left[-(2j-1) + \sqrt{1 + \frac{2}{\nu}}\right], \tag{23}$$

F denotes the hypergeometric function and $e_j - t = j - 1$. In particular, when $\nu \gg 1$ then $t \sim \frac{1}{2\nu} \ll 1$ and the spectrum of h consists of only one eigenvalue

$$\mu_1 = -\frac{\nu^2}{4}\left[-1 + \sqrt{1 + \frac{2}{\nu}}\right]^2 \sim -\frac{1}{4} + O(\nu^{-1}) \tag{24}$$

with the associated normalized eigenvector

$$\varphi_1(x) = \frac{\tilde{\varphi}_1(x)}{\|\tilde{\varphi}_1(x)\|_{L^2(\mathbb{R})}}, \qquad \tilde{\varphi}_1(x) = [1 - \tanh^2(\nu x)]^{e_1/2}. \tag{25}$$

We recall that the absolute continuous spectrum of the operators

$$h_\nu = -\partial_x^2 + V_\nu(x) \qquad \text{and} \qquad h_\infty = -\partial_x^2 - \delta,$$

where δ denotes the Dirac's delta at $x = 0$, coincides with the positive real axis,

$$\sigma_{ac}(h_\nu) = \sigma_{ac}(h_\infty) = [0, +\infty),$$

and that, see e.g. [1, theorem 3.2.3], $h_\nu \to h_\infty$ as $\nu \to +\infty$ in the norm resolvent sense:

$$\lim_{\nu\to\infty} \|r_\nu(\zeta) - r_\infty(\zeta)\| = 0, \qquad \operatorname{Im}\zeta > 0,$$

where $r_\nu(\zeta) = [\zeta - h_\nu]^{-1}$ and $r_\infty(\zeta) = [\zeta - h_\infty]^{-1}$. Furthermore, making use of the same arguments as in [1, section 3.2], it follows that for any rapidly decreasing test function φ

$$\lim_{\nu\to\infty} \langle \varphi, [r_\nu(\zeta) - r_\infty(\zeta)]\varphi\rangle_{L^2(\mathbb{R})} = 0, \qquad \operatorname{Im}\zeta \geqslant 0, \tag{26}$$

uniformly for ζ belonging to a compact set $[a, b] \times \mathrm{i}[0, c]$ for any $a, b, c > 0$.

Computation of the coefficient a. For $\nu \to \infty$ we have

$$\sigma_d(h_\nu) = \left\{\mu_1 = -\tfrac{1}{4} + O(\nu^{-1})\right\}. \tag{27}$$

We take ν large enough so that for the set $\Sigma = \{E = E_{j,n} = \mu_j + E_n\}$ of eigenvalues of $H_0^{V_\nu}$ holds

$$E_{1,1} = E_1 - \tfrac{1}{4} + O(\nu^{-1}) < E_1 < E_{1,2} = E_2 - \tfrac{1}{4} + O(\nu^{-1}) < E_2. \tag{28}$$

Then we apply the perturbative theory to the embedded eigenvalue

$$E = E_{1,2} = E_2 + \mu_1 \tag{29}$$

with the associated eigenvector

$$\psi(x, y, z) = \varphi_1(x)\chi_2(y, z). \tag{30}$$

In such a case $k^\star = 2$ and

$$\operatorname{Im} a_2 = -\lim_{\rho\to 0^+} \operatorname{Im}\left\{\sum_{k=1,2} \langle d_k, r_\nu(E - E_k - \mathrm{i}\rho)d_k\rangle_{L^2(\mathbb{R})} - \left|\left\langle \psi, U_\varepsilon^{V_\nu}\psi\right\rangle_{L^2(\Omega)}\right|^2 (\mathrm{i}\rho)^{-1}\right\} \tag{31}$$

where $r_\nu(\zeta)$ is the resolvent of h_ν and

$$d_k(x) = \left\langle \chi_k, U_\varepsilon^{V_\nu}\psi\right\rangle_{L^2(\omega)}. \tag{32}$$

An integration by parts shows that $\langle\chi_2, \partial_\tau\chi_2\rangle_{L^2(\omega)} = 0$. We thus get

$$\begin{aligned}\left\langle \psi, U_\varepsilon^{V_\nu}\psi\right\rangle_{L^2(\Omega)} &= -2\varepsilon\langle\varphi_1, \dot\alpha\partial_x\varphi_1\rangle_{L^2(\mathbb{R})}\langle\chi_2, \partial_\tau\chi_2\rangle_{L^2(\omega)} - \varepsilon\langle\varphi_1, \ddot\alpha\varphi_1\rangle_{L^2(\mathbb{R})}\langle\chi_2, \partial_\tau\chi_2\rangle_{L^2(\omega)} \\ &\quad - \varepsilon^2\langle\varphi_1, \dot\alpha^2\varphi_1\rangle_{L^2(\mathbb{R})}\left\langle\chi_2, \partial_\tau^2\chi_2\right\rangle_{L^2(\omega)} \\ &= -C_0\varepsilon^2,\end{aligned}$$

where

$$C_0 = \langle\varphi_1, \dot\alpha^2\varphi_1\rangle_{L^2(\mathbb{R})}\left\langle\chi_2, \partial_\tau^2\chi_2\right\rangle_{L^2(\omega)} = \left\langle\chi_2, \partial_\tau^2\chi_2\right\rangle_{L^2(\omega)}.$$

Furthermore, from (32) we obtain that

$$d_1(x) = -2\varepsilon C_1\partial_x\varphi_1 - \varepsilon^2 C_2\varphi_1, \qquad d_2(x) = -\varepsilon^2 C_0\varphi_1 \tag{33}$$

where

$$C_1 = \langle\chi_1, \partial_\tau\chi_2\rangle_{L^2(\omega)} \qquad \text{and} \qquad C_2 = \left\langle\chi_1, \partial_\tau^2\chi_2\right\rangle_{L^2(\omega)}.$$

Collecting all these facts and keeping in mind that $E = E_2 + \mu_1$ and φ_1 is the eigenfunction of h_ν with eigenvalue μ_1 we get, after some tedious, but straightforward calculations, that

$$\lim_{\rho\to 0+} \operatorname{Im}\left\{\langle d_2, r_\nu(E - E_2 - \mathrm{i}\rho)d_2\rangle_{L^2(\mathbb{R})} - \left|\left\langle\psi, U_\varepsilon^{V_\nu}\psi\right\rangle\right|^2_{L^2(\Omega)}(i\rho)^{-1}\right\} = 0.$$

This implies

$$\operatorname{Im} a_2 = -\lim_{\rho\to 0^+}\operatorname{Im}\langle d_1, r_\nu(E - E_1 - \mathrm{i}\rho)d_1\rangle_{L^2(\mathbb{R})} \tag{34}$$

$$= -\lim_{\rho\to 0^+}\operatorname{Im}[4\varepsilon^2|C_1|^2\langle\partial_x\varphi_1, r_\nu(E - E_1 - \mathrm{i}\rho)\partial_x\varphi_1\rangle] + O(\varepsilon^3). \tag{35}$$

We now pass to the limit $\nu \to \infty$ which implies

$$\mu_1 \to -\tfrac{1}{4}, \qquad \varphi_1 \to \phi = \sqrt{\tfrac{1}{2}}\,\mathrm{e}^{-|x|/2}, \qquad h_\nu \to h_\infty, \tag{36}$$

where the last limit is reached in the norm resolvent sense. We recall also that the resolvent $[\zeta - h_\infty]^{-1}$ has the kernel given by

$$\mathcal{K}_\zeta(x, x') = \mathcal{K}_\zeta^0(x, x') + \mathcal{K}_\zeta^1(x, x'),$$

where

$$\mathcal{K}_\zeta^0(x, x') = \frac{1}{2k\mathrm{i}}\,\mathrm{e}^{\mathrm{i}k|x-x'|}, \qquad \mathcal{K}_\zeta^1(x, x') = -\frac{1}{2k}\frac{1}{2k+\mathrm{i}}\,\mathrm{e}^{\mathrm{i}k[|x|+|x'|]}, \qquad \zeta = k^2, \quad \operatorname{Im} k > 0$$

(see [1]). In our case

$$\zeta = k^2 = E - E_1 - \mathrm{i}\rho = E_2 - E_1 + \mu_1 - \mathrm{i}\rho. \tag{37}$$

We will denote by $\mathcal{K}_\zeta^0$ and $\mathcal{K}_\zeta^1$ the integral operators with the kernels $\mathcal{K}_\zeta^0(x,x')$ and $\mathcal{K}_\zeta^1(x,x')$, respectively. Note that $\partial_x\varphi_1(x)$ is an odd function, which implies that $\mathcal{K}_\zeta^1\partial_x\varphi_1 \equiv 0$. Now $h_\nu \to h_\infty$ as in (26). Since $E - E_1$ is not an eigenvalue of h_ν for any ν large enough (in fact $E - E_1$ belongs to the absolute continuous spectrum of the operators h_ν and h_∞) and d_1 is an exponentially decreasing function as $|x| \to \infty$, we can pass to the limit $\nu \to \infty$ replacing $r_\nu(E - E_1 - \mathrm{i}\rho)$ on the rhs of (34) by $\mathcal{K}_\zeta^0$ and φ_1 by ϕ:

$$\begin{aligned}\lim_{\nu\to\infty} \operatorname{Im} a_2 &= -\lim_{\nu\to\infty}\lim_{\rho\to 0^+} \operatorname{Im}\langle d_1, r_\nu(E - E_1 - \mathrm{i}\rho)d_1\rangle_{L^2(\mathbb{R})}\\ &= -\lim_{\rho\to 0^+} \operatorname{Im}[4\varepsilon^2|C_1|^2\langle\partial_x\phi, r_\infty(E - E_1 - \mathrm{i}\rho)\partial_x\phi\rangle] + O(\varepsilon^3),\end{aligned} \tag{38}$$

where the remainder term is uniform with respect to ρ. An explicit computation then gives

$$\lim_{\rho\to 0^+} \operatorname{Im}\big\langle\partial_x\phi, \mathcal{K}_\zeta^0\partial_x\phi\big\rangle = \frac{4\sqrt{E_2 - E_1 + \mu_1}}{[1 + 4(E_2 - E_1 + \mu_1)]^2}. \tag{39}$$

In view of (36) and (39) we get

$$\lim_{\nu\to\infty} \operatorname{Im} a_2 = -\varepsilon^2|C_1|^2\frac{\sqrt{E_2 - E_1 - \frac{1}{4}}}{(E_2 - E_1)^2}.$$

The proof is complete.

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